

## Classical Mechanics Graduate Exam

December 15, 2004

NOTE: Do NOT write your name on any paper you hand in. Only write the selected number for the subsequent identification, so that the problems may be graded anonymously.

Show all work. Justify your answers.

1. At relative distances  $r > R$ , particles  $A$  of mass  $m_1$  are repelled from particles  $B$  of mass  $m_2$  with central forces  $F = k/r^{5/2}$ . At distances  $r < R$ , inelastic processes set in, altering intrinsic structure of the particles  $A$  and  $B$ .
  - (a) Find the potential  $V$  for the central force  $F$ .
  - (b) Particle  $A$  is directed from far away towards particle  $B$  at rest. What is the minimum initial momentum  $p_1$  of  $A$  needed for the inelastic processes to set in?
  - (c) Find the cross section for inelastic processes, as a function of  $p_1$ .
  
2. (a) Demonstrate that the transformation

$$\begin{aligned} x &= X \cos \lambda + P_Y \sin \lambda, & y &= Y \cos \lambda + P_X \sin \lambda, \\ p_x &= P_X \cos \lambda - Y \sin \lambda, & p_y &= P_Y \cos \lambda - X \sin \lambda, \end{aligned}$$

where  $\lambda$  is an arbitrary parameter, is canonical.

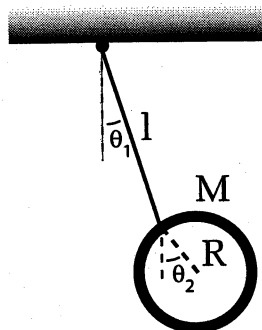
- (b) Assuming the Hamiltonian of a two-dimensional harmonic oscillator:

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2},$$

find the Hamiltonian after the transformation (a), in terms of  $X, Y, P_X$  and  $P_Y$ .

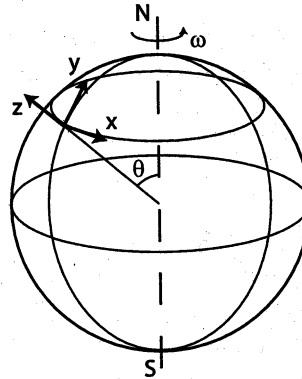
- (c) What is the Hamiltonian for a particle of mass  $m = 1$  and charge  $q = 1$ , confined to the  $x$ - $y$  plane, moving under combined influence of the harmonic oscillator potential  $V = (x^2 + y^2)/2$  and the uniform magnetic field  $\vec{B}$  in  $z$ -direction, represented by the vector potential  $\vec{A} = (0, Bx, 0)$ ?
- (d) Employ the transformation from (a) to the Hamiltonian from (c) and find the condition on  $\lambda$  under which the transformed Hamiltonian becomes that of an anisotropic harmonic oscillator.

3. A thin ring of mass  $M$  and radius  $R$  hangs from a rigid support by a thin thread of length  $\ell$ .

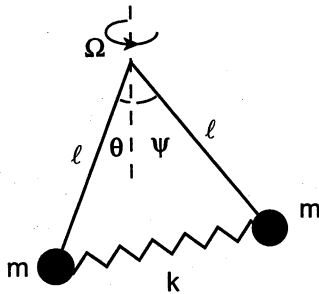


- (a) Write the kinetic and potential energies for the motion of the ring within the plane of the figure.
- (b) Expand the kinetic and potential energies in  $\theta_1$  and  $\theta_2$  by considering small deviations of the ring from its equilibrium position.
- (c) Find the normal modes of oscillation of the ring within the plane, frequencies and amplitude vectors. Discuss those modes.

4. A particle is projected vertically upward from a location at colatitude  $\theta$  on the Earth's surface. If it is intended that the particle falls back to the ground at a distance no greater than  $d$  from the launch point on the ground, what is the maximum elevation  $h$  above the ground to which the particle can rise? Neglect air resistance and obtain the answer to the leading order in the Earth's angular velocity  $\omega$ . In which geographic direction from the launch point will the particle land?



5. The governor of a steam engine consists of two small balls of mass  $m$  that are mounted on light rods of length  $\ell$ . At their other ends, the rods are attached, through hinges, to a vertical axis. The plane of the rods rotates at constant angular velocity  $\Omega$  about the axis. A spring of spring constant  $k$  connects the two governor's balls. Below, assume that the rods are massless, balls are of negligible size and neutral length of the spring is negligible. Use the angles  $\theta$  and  $\psi$  as generalized coordinates.



- Find the kinetic and potential energies and the Lagrangian for the governor.
- Obtain the Lagrange equations.
- Find equilibrium points for the governor. Note: those points are characterized by symmetry  $\theta = \psi$ .
- Obtain the Hamiltonian for the governor.
- Is the Hamiltonian conserved? Is the energy conserved? Is the angular momentum conserved? Explain why.