PHY-820 - CLASSICAL MECHANICS

Final Exam

Problem 1 - 10 Points

A homogeneous cube of side length l and mass m is initially at rest in unstable equilibrium standing on one of its edges touching the horizontal plane. It then begins to tip over from this position.

a) Determine the moment of inertia of the cube around an axis parallel to an edge and through its center of mass. Remember that the elements of the tensor are given by $I_{ij} = \int_V dV \rho(\vec{r}) \cdot (\vec{r}^2 \delta_{ij} - r_i r_j)$ [2 pts]

b) Using symmetry arguments, determine the entire inertia tensor [2 pts]

c) Assume the edge in contact with the plane cannot slide. Determine the moment of inertia around that edge. What is the angular velocity at the time the face of the cube contacts the plane? [3 pts]

d) Assume the plane is frictionless so that the edge in contact with the plane can slide, and so that the rotation occurs around the cube's center axis. What is the angular velocity at the time the face of the cube contacts the plane? [3 pts]

Problem 2 - 10 Points

Consider the 2-dimensional motion of a particle moving in an attractive central force $F(r) = -k/r^{\alpha}$, where $2 < \alpha < 3$.

a) Determine the Lagrangian and the equations of motion in polar coordinates. [2 pts]

b) Determine the generalized momenta and the Hamiltonian. [2 pts]

c) Show that one of the generalized momenta is conserved. Is the Hamiltonian conserved? [2 pts]

d) Utilizing that one of the momenta is conserved, re-write the equations of motion of the other variable so that it becomes one dimensional. [2 pts]

e) Without attempting to solve the problem, qualitatively classify the various cases of solutions depending on the energy of the system. A sketch may be helpful. [2 pts]

Problem 3 - 10 Points

Consider a double pendulum swinging in the plane as follows. A uniform rod of length $2a$ and mass $2m$ is attached to pivot around one of its ends. At the other end of the rod, a string of length l is attached, at the end of which there is a point mass m.

a) Determine the Lagrangian of the system in terms of the angles θ and ϕ between the rod and string and the vertical, respectively, keeping terms up to second order in the variables. [6 pts]

b) Determine the equations of motion for $\ddot{\theta}$ and $\ddot{\phi}$ [4 pts]
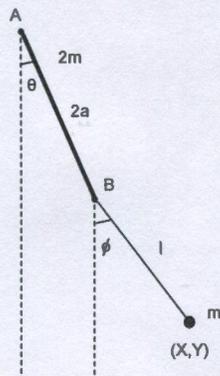
Problem 4 - 10 Points

A thin uniform rod of mass M and length $2b$ is placed on top of a stationary cylinder of radius a, so that the center of mass of the rod rests on the uppermost part of the cylinder. The cylinder is attached to the ground and does not move. On both ends of the rod, masses m are attached by strings of lengths l_1 and l_2. The setup is then moved from equilibrium to perform small oscillations. The contact between the cylinder and the rod is assumed to be dominated by friction, i.e. the rod does not slide over the cylinder's surface.

a) Determine the Cartesian position of the center of mass of the rod as a function of its tilt angle θ. Also determine the Cartesian positions of the two masses as a function of the angles θ_1 and θ_2 of their respective strings. [3 pts]

b) Determine the Cartesian velocities of the three locations in part a) as a function of θ, θ_1 and θ_2. [2 pts]

c) Perform a small angle approximation of the velocities and determine the Lagrangian under this approximation. [3 pts]

d) Determine the resulting equations of motion and the frequencies of oscillations. [2 pts]

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