Vector Formulas

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi=0$$

$$\nabla\cdot(\nabla\times\mathbf{a})=0$$

$$\nabla\times(\nabla\times\mathbf{a})=\nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a})=\mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a})=\nabla\psi\times\mathbf{a}+\psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b})=(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}(\nabla\cdot\mathbf{b})-\mathbf{b}(\nabla\cdot\mathbf{a})+(\mathbf{b}\cdot\nabla)\mathbf{a}-(\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r=|\mathbf{x}|$, and $\mathbf{n}=\mathbf{x}/r$ is a unit radial vector, then

$$\nabla\cdot\mathbf{x}=3 \qquad \nabla\times\mathbf{x}=0$$

$$\nabla\cdot\mathbf{n}=\frac{2}{r} \qquad \nabla\times\mathbf{n}=0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n}=\frac{1}{r}[\mathbf{a}-\mathbf{n}(\mathbf{a}\cdot\mathbf{n})]=\frac{\mathbf{a}_\perp}{r}$$

VECTOR CALCULUS

Cylindrical coordinates (z, r, α):

$$\nabla S = \frac{\partial S}{\partial z}\mathbf{e}^{(z)} + \frac{\partial S}{\partial r}\mathbf{e}^{(r)} + \frac{1}{r}\frac{\partial S}{\partial \alpha}\mathbf{e}^{(\alpha)}; \tag{1}$$

$$\operatorname{div}\mathbf{v} = \frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\alpha}{\partial \alpha}; \tag{2}$$

$$\begin{aligned}\operatorname{curl}\mathbf{v} \;=\; & \frac{1}{r}\Big[\frac{\partial}{\partial r}(r v_\alpha) - \frac{\partial v_r}{\partial \alpha}\Big]\mathbf{e}^{(z)} \\ + \; & \Big[\frac{1}{r}\frac{\partial v_z}{\partial \alpha} - \frac{\partial v_\alpha}{\partial z}\Big]\mathbf{e}^{(r)} \\ + \; & \Big[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\Big]\mathbf{e}^{(\alpha)};\end{aligned} \tag{3}$$

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \alpha^2}. \tag{4}$$

Spherical polar coordinates (r, θ, α):

$$\nabla S = \frac{\partial S}{\partial r}\mathbf{e}^{(r)} + \frac{1}{r}\frac{\partial S}{\partial \theta}\mathbf{e}^{(\theta)} + \frac{1}{r\sin\theta}\frac{\partial S}{\partial \alpha}\mathbf{e}^{(\alpha)}; \tag{5}$$

$$\operatorname{div}\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\alpha}{\partial \alpha}; \tag{6}$$

$$\begin{aligned}\operatorname{curl}\mathbf{v} \;=\; & \frac{1}{r\sin\theta}\Big[\frac{\partial}{\partial \theta}(\sin\theta\, v_\alpha) - \frac{\partial v_\theta}{\partial \alpha}\Big]\mathbf{e}^{(r)} \\ + \; & \frac{1}{r}\Big[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \alpha} - \frac{\partial}{\partial r}(r v_\alpha)\Big]\mathbf{e}^{(\theta)} \\ + \; & \frac{1}{r}\Big[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\Big]\mathbf{e}^{(\alpha)};\end{aligned} \tag{7}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\Big(\sin\theta\frac{\partial}{\partial \theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \alpha^2}. \tag{8}$$

Expressions Involving $\sqrt{x^2 \pm a^2}$ or $\sqrt{a^2 - x^2}$

126. $\int \sqrt{x^2 \pm a^2}\, dx = \frac{1}{2}[x\sqrt{x^2 \pm a^2} \pm a^2 \log(x + \sqrt{x^2 \pm a^2})]$

127. $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right)$

128. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log(x + \sqrt{x^2 + a^2}), \quad \text{or} \quad \sinh^{-1}\frac{x}{a}$

129. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log(x + \sqrt{x^2 - a^2}), \quad \text{or} \quad \cosh^{-1}\frac{x}{a}$

130. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}, \quad \text{or} \quad -\cos^{-1}\frac{x}{a}$

131. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\cos^{-1}\frac{a}{x}$

132. $\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right)$

133. $\int \frac{\sqrt{a^2 \pm x^2}}{x}\, dx = \sqrt{a^2 \pm x^2} - a \log\frac{a + \sqrt{a^2 \pm x^2}}{x}$

134. $\int \frac{\sqrt{x^2 - a^2}}{x}\, dx = \sqrt{x^2 - a^2} - a \cos^{-1}\frac{a}{x}$

135. $\int \frac{x\, dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2}$

136. $\int \frac{x\, dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}$

*See Formulas 703–704.

137. $\int x\sqrt{x^2 \pm a^2}\, dx = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3}$

138. $\int x\sqrt{a^2 - x^2}\, dx = -\frac{1}{3}\sqrt{(a^2 - x^2)^3}$

139. $\int \sqrt{(x^2 \pm a^2)^3}\, dx$
$= \frac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2 \pm a^2} + \frac{3a^4}{2}\log(x + \sqrt{x^2 \pm a^2})\right]$

140. $\int \sqrt{(a^2 - x^2)^3}\, dx$
$= \frac{1}{4}\left[x\sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2 - x^2} + \frac{3a^4}{2}\sin^{-1}\frac{x}{a}\right]$

141. $\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$

142. $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}}$

143. $\int \frac{x\, dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}$

144. $\int \frac{x\, dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}$

$\int_0^{2\pi}\left(\frac{1 - \cos x}{x}\right)dx \simeq 2.44$

145. $\int x\sqrt{(x^2 \pm a^2)^3}\, dx = \frac{1}{5}\sqrt{(x^2 \pm a^2)^5}$

146. $\int x\sqrt{(a^2 - x^2)^3}\, dx = -\frac{1}{5}\sqrt{(a^2 - x^2)^5}$

*See Formulas 703–704.

Harmonic

Spherical $_\wedge$ functions

$Y_{lm}(\theta,\varphi)$

$l\ m$	$Y_{lm}(\theta\varphi)$	
0 0	$\frac{1}{2\sqrt{\pi}}$ →	$\frac{1}{2\sqrt{\pi}}$
1 0	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$ →	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$
1 ±1	$\mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{\pm i\varphi}$ →	$\mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{\pm i\varphi}$
2 0	$\frac{1}{4}\sqrt{\frac{5}{\pi}}(2\cos^2\theta - \sin^2\theta)$ →	$\frac{1}{4}\sqrt{\frac{5}{\pi}}\left(2\cos^2\theta - \sin^2\theta\right)$
2 ±1	$\mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\cos\theta\sin\theta e^{\pm i\varphi}$	
2 ±2	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{\pm 2i\varphi}$	
3 0	$\frac{1}{4}\sqrt{\frac{7}{\pi}}(2\cos^3\theta - 3\cos\theta\sin^2\theta)$	
3 ±1	$\mp\frac{1}{8}\sqrt{\frac{21}{\pi}}(4\cos^2\theta\sin\theta - \sin^3\theta)e^{\pm i\varphi}$	
3 ±2	$\frac{1}{4}\sqrt{\frac{105}{2\pi}}\cos\theta\sin^2\theta e^{\pm 2i\varphi}$	
3 ±3	$\mp\frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta e^{\pm 3i\varphi}$	

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0}(\theta)$$

$$Y^*_{lm}(\vec{n}) = (-)^m Y_{l,-m}(\vec{n})$$

Orthogonality:
$$\int d\Omega_n\, Y^*_{l'm'}(\vec{n})\, Y_{lm}(\vec{n}) = \delta_{l'l}\,\delta_{m'm} \quad ;$$

$$\int_{-1}^{1} d\eta\, P_{l'}(\eta)\, P_l(\eta) = \frac{2}{2l+1}\,\delta_{l'l} \quad ;$$

$$\frac{dP_{l+1}(x)}{dx} - \frac{dP_{l-1}(x)}{dx} = (2l+1)P_l(x) \quad \text{for } l \geq 1$$

Addition theorem:
$$P_l(\cos\chi_{\vec{n}\vec{n}'}) = \sum_{m=-l}^{l} \frac{4\pi}{2l+1}\, Y^*_{lm}(\vec{n})\, Y_{lm}(\vec{n}')$$

Multipole expansion (generating function): $\vec{r} = r\vec{n}$, $\vec{r}\,' = r'\vec{n}'$,
$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}}\, P_l(\cos\chi_{\vec{n}\vec{n}'})$$

$P_l(1) = 1$, $P_l(-1) = (-1)^l$, $P_l(0) = \frac{1+(-1)^l}{2}(-1)^{l/2}\frac{l!}{2^l\,[(l/2)!]^2}$

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Subject Exam (Total: 50 points)
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Problem 1. A point charge q is brought to a position at a distance d away from an infinite plane conductor held at zero potential.
(5 points)(a) Find the surface-charge density induced on the plane, and plot it.
(3 points)(b) Find the total force acting on the plane.
(2 points)(c) Find the work needed to remove the charge q from its position to infinity.

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Problem 2. A spherical surface of radius R has charge uniformly distributed over its surface with a density σ, except for a spherical cap at the north pole. The spherical cap is defined by the cone with polar angle $\theta = \alpha$, and it does not carry any charge.
(3 points)(a) Find the potential inside the spherical surface.
(3 points)(b) Find the potential outside the spherical surface.
(2 points)(c) Find the magnitude and the direction of the electric field at the origin.
(2 points)(d) Find the potential and the electric field inside the spherical surface as the spherical cap becomes very small. You only need to keep the leading term, in the expansion of α, for this part of problem.

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Problem 3. A non-conducting sphere of radius a carries a uniform surface charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω.
(2 points)(a) Find the total charge Q and the current density $\vec{J}$.
(5 points)(b) Find the vector potential both inside and outside the sphere.
(3 points)(c) Find the magnitude and the direction of the magnetic field both inside and outside the sphere.

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Problem 4. A thin linear antenna of length d is excited in such a way that the full wavelength (λ) of oscillation of the sinusoidal current is equal to d. Denote the time-dependent current density as $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$. As usual, the real part of such expression is to be taken to obtain physical quantities.

(2 points)(a) Write down the current density $\vec{J}(\vec{x})$ as a function of position $\vec{x}$ in 3-dimensional space.

(3 points)(b) Find the vector potential $\vec{A}(\vec{x})$ induced by radiation in the radiation zone, i.e. at a distance much larger than λ.

(3 points)(c) Calculate the power radiated per unit solid angle, and plot the angular distribution of radiation.

(2 points)(d) Find the total power radiated.

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Problem 5. Consider an elliptically polarized electromagnetic wave whose electric component is given by

$$\vec{E}(z,t) = \hat{\imath}E_0 \sin[\omega(t - z/c)] + \hat{\jmath}E_0 \sin[\omega(t - z/c) + \pi/4], \tag{1}$$

where ω is the angular frequency of the wave and c is the speed of wave.
(2 points)(a) Find the magnetic component of the electromagnetic wave.
(2 points)(b) Calculate the energy density of the electromagnetic wave propagating through free space.
(1 point)(c) Find the speed with which the energy is propagating.
(5 points)(d) Find the smallest and largest value of the Poynting vector for this wave.