Number ....

QUANTUM MECHANICS
Subject Exam

Total $4 \times 20 + 5 \times 4 = 100$ points

April 29, 2013

PROBLEM 1. A beam of particles of spin 1/2 polarized along the unit vector **n** characterized by the polar angle θ and azimuthal angle φ is transmitted through an analyzer that lets through only the particles polarized along the direction $\mathbf{n}'(\theta', \varphi')$. Find the fraction of the transmitted intensity.

PROBLEM 2. A particle of mass m moves in the potential well

$$U(x) = g|x|, \quad g > 0. \tag{6}$$

Using an approximate method of your choice estimate the ground state energy for a particle of mass m in this well. The exact result is

$$E_0 = 0.809 \left(\frac{\hbar^2 g^2}{m} \right)^{1/3}. \tag{7}$$

PROBLEM 3. An electron is confined inside an empty sphere of radius R. Determine the pressure exerted on the surface of the sphere if the electron is in the lowest s-state.

PROBLEM 4. A pulse of the electric field,

$$\mathcal{E}(t) = \mathcal{E}_0 e^{-(t/\tau)^2}, \tag{23}$$

is applied along the oscillation line to a particle originally in the ground state of the one-dimensional harmonic potential. Using the perturbation theory, find the probability for the particle to be in an excited state after the pulse. Compare the efficiency of energy transfer for two pulses supplying the same total momentum but differing in the value of the parameter τ if both pulses justify the use of perturbation theory. */It might be convenient to express the perturbation in terms of the oscillator creation and annihilation operators./*

5. QUICK QUESTIONS

5.1. Estimate the lifetime of the $2p$ level in the hydrogen atom and the corresponding wavelength for the transition to the ground state (*give numbers!*).

5.2. The electrons emitted from an atomic process are described by the wave function

$$\psi(\mathbf{r}) = \frac{e^{ikr}}{r}(\hat{\mathbf{s}} \cdot \mathbf{n}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{31}$$

where $\hat{\mathbf{s}}$ is the spin operator, $\mathbf{n} = \mathbf{r}/r$ the unit vector, and the column is the spinor in the z-representation. Determine the quantum numbers of this state (orbital momentum ℓ, total angular momentum j, its projection j_z, parity) and the angular distribution of emitted electrons.

5.3. The lowest configuration of nucleons in the nucleus of the calcium isotope ^{42}Ca consists of the closed-shell core ^{40}Ca and two valence neutrons on the orbital $f_{7/2}$. What are the possible values of the total angular momentum for this configuration?

5.4. For the following operators find out if they correspond to conserved quantities for free motion of a relativistic particle of spin 1/2:

a. orbital momentum squared $\hat{\vec{\ell}}^{\,2}$,
b. spin squared $\hat{\mathrm{s}}^2$,
c. helicity $\hat{h} = (\hat{\mathrm{p}} \cdot \vec{\Sigma})$,
d. spatial inversion $\hat{P}$ acting according to $\mathbf{r} \to -\mathbf{r}$,
e. operator $\beta\hat{P}$ (spatial inversion multiplied by the Dirac matrix β).

5.5. Find the magnetic moment of the atomic term $^6G_{3/2}$.

Number ....

QUANTUM MECHANICS

Subject Exam

Total $20 \times 5 = 100$ points

April 30, 2012

PROBLEM 1. *Do particle fall on the Earth?*
Consider a neutral particle of mass m in the local gravitational field of the Earth treated as an impenetrable flat surface. Using one of the approximate approaches, estimate the mean height above the ground for the lowest quantum-mechanical state of the particle. (Give numbers for the neutron and for a tennis ball.)

PROBLEM 2. Consider a system of two different particles with spin 1/2 each. The spins interact through the Hamiltonian

$$\hat{H}_0 = A(\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2). \tag{5}$$

a. Find the stationary states of the system and their energies.

b. In addition, the static magnetic field $\vec{B}$ is applied to the system. Find the stationary states of the system and their energies if the spin gyromagnetic ratios of the particles are equal to g_1 and g_2.

PROBLEM 3. A particle of mass m and electric charge e is placed in the static uniform magnetic field $B = B_z$ and static uniform electric field $\mathcal{E} = \mathcal{E}_x$. Find the energy spectrum and the stationary wave functions. Are the energy eigenvalues degenerate?

PROBLEM 4. Using the Born approximation, calculate the differential cross section for scattering of a fast particle by the Yukawa potential,

$$U(r) = \frac{g}{r} e^{-\mu r}. \tag{20}$$

Explain the limit $\mu \to 0$. Formulate the condition of validity of the Born approximation. Is the total cross section finite?

5. QUICK QUESTIONS.

1. The wave function of a particle of spin 1/2 is given by

$$\psi = (\vec{\sigma} \cdot \mathbf{r}) f(r) \chi, \tag{25}$$

where χ is some spinor. Find the quantum numbers (ℓ, j, parity) of this state.

2. Find the magnetic moment of the atom in the state $^6G_{3/2}$.

3. A quantum level of a system with quantum numbers J, M of total angular momentum is split by the weak static uniform field along the quantization axis. Explain the difference (if any) in the character of splitting between the cases of electric and magnetic field.

4. Estimate by the order of magnitude the lifetime of the $2p$ level in the hydrogen atom.

5. What is the velocity operator for a Dirac particle and what are its eigenvalues?

Number ....

QUANTUM MECHANICS

Subject Exam

Total $4 \times 20 + 5 \times 4 = 100$ points

August 29, 2011

1. An electron with energy E moves over a metallic strip of width a that can be modeled by a potential well of depth W.

   a. Find the reflection and transmission coefficients.

   b. Find the energy values corresponding to the full transmission (*resonances*) and explain the wave mechanism of this phenomenon.

   c. For given values of E and W find the width a that gives the maximum reflection coefficient and explain the wave mechanism for this case.

2. Consider a particle of mass m in the one-dimensional harmonic oscillator field of frequency ω.

   a. Write down equations of motion for Heisenberg operators $\hat{x}(t)$ and $\hat{p}(t)$.

   b. Solve these differential equations with the initial conditions $\hat{x}(0) = \hat{x}$, $\hat{p}(0) = \hat{p}$.

   c. Find the commutators

$$[\hat{x}(t), \hat{x}(t')], \quad [\hat{x}(t), \hat{p}(t')], \quad [\hat{p}(t), \hat{p}(t')]$$

   for arbitrary time moments t and t' and determine the corresponding uncertainty relations. Can $\hat{x}(t)$ and $\hat{p}(t')$ have simultaneously certain values?

3. a. Using the Born approximation, find the scattering amplitude and the differential cross section for scattering of a particle of mass m by the Yukawa potential

$$U(r) = \frac{g}{r} e^{-r/R}.$$

b. Derive the expression for the total (integrated over angles) cross section $\sigma(E)$. Find σ in limiting cases of low energy, $kR \ll 1$, and high energy, $kR \gg 1$.

c. Formulate the criteria of validity of the Born approximation in both limits.

d. Discuss the limit of $R \to \infty$.

4. Two particles of equal masses interact through the contact potential

$$U = g\delta(\mathbf{r}_1 - \mathbf{r}_2).$$

The particles are placed in an impenetrable box with sides $a > b > c$, but $a < (3/2)b$. Using symmetry arguments, construct the wave functions of the ground state and the first excited state of the system in terms of the single-particle functions in the box. Assume that the particles are

a. different;
b. identical with spin $s = 0$;
c. identical with spin $s = 1/2$ and total spin $S = 0$;
d. identical with spin $s = 1/2$ and total spin $S = 1$.

You do not have to calculate perturbed energies but you will get extra 5 points if you do it in the linear approximation with respect to the strength parameter g; here the integral might be useful,

$$\int_0^a dx \, \sin^4\left(\frac{n\pi x}{a}\right) = \frac{3a}{8}, \quad n = 1, 2, \ldots$$

5. QUICK QUESTIONS

a. For the lowest electron orbit in the hydrogen atom, estimate magnitude of the electric field of the nucleus at the orbit (in V/cm).

b. For two possible states of the same particle,

$$\psi_1(x) = \exp(ikx - \alpha x^2)$$

and

$$\psi_2(x) = \exp(ikx - 2\alpha x^2),$$

where k and $\alpha > 0$ are real constants, what is the ratio $\langle p_x \rangle_1 / \langle p_x \rangle_2$ of the expectation values of the momentum?

c. A quantum system has, among its constants of motion, the momentum component p_x and the total angular momentum component J_z. Is it possible to conclude from this fact that p_y and p_z are also conserved quantities?

d. Determine the orbital momentum and parity quantum numbers if the wave function of a particle has the form ($r^2 = x^2 + y^2 + z^2$)

$$\psi(x, y, z) = x\, y\, z\, e^{-r^2/a^2}.$$

e. Let $\Psi(\mathbf{r}, t)$ be a solution of the Dirac equation. What is the charge conjugate function Ψ_C? What equation is satisfied by Ψ_C?

Number ....

QUANTUM MECHANICS
Subject Exam
Total $20 \times 5 = 100$ points
May 2, 2011

1. The fine structure splitting of atomic levels is determined by the spin-orbit interaction described by the Hamiltonian

$$H'_{\text{f.s.}} = A(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}), \tag{1}$$

where $\mathbf{L}$ and $\mathbf{S}$ are the total orbital momentum and the total spin of the electrons, respectively, and A is the coupling constant.

a. Find if the components of the vectors $\mathbf{L}$, $\mathbf{S}$ and $\mathbf{L} \pm \mathbf{S}$ are conserved. Are their squares $\mathbf{L}^2, \mathbf{S}^2, (\mathbf{L} \pm \mathbf{S})^2$ conserved?

b. For an atom in the unperturbed state ^5F find the number of fine structure components and the energy spacing between them.

c. Describe the additional splitting of fine structure levels in a weak static magnetic field. An experiment shows that one of the fine structure levels is not split. Determine the total angular momentum J of this level.

2. A particle is moving in the magnetic trap formed by a uniform magnetic field $B = B_z$ and two one-dimensional harmonic oscillator fields $U(x, z) = (1/2)m[\omega_x^2 x^2 + \omega_z^2 z^2]$. Find the energy spectrum of the stationary states. Are the levels degenerate?

3. A particle of mass m is freely moving inside a sphere with an impenetrable surface shell at $r = a$, $U(r) = 0$ for $r < a$.

a. Find the normalized wave functions and the energy spectrum of the s-wave states.

b. Find the energy of the ground state for the electron if $a = 10^{-10}$ m.

c. Show that the radial function $R_{\ell=1}(r)$ for the p-wave in the same potential can be found as $R_{\ell=1} = dR_{\ell=0}/dr$ and derive the equation that defines the energy spectrum of p-wave states.

4. An exotic oxygen nucleus ^{19}O with 8 protons and 11 neutrons can be presented as a closed-shell core ^{16}O with 8 protons and 8 neutrons and three valence neutrons which can occupy $d_{5/2}$ and $s_{1/2}$ orbits. Find the possible values of the total angular momentum of the system and check the total number of allowed states for this configuration.

5. **QUICK QUESTIONS**

a. For a given one-dimensional barrier $U(x)$, what is the relation between the reflection coefficients R_l and R_r for incident waves of the same energy coming from the left and from the right, respectively?

b. Why is the magnetic moment of the electron measured with amazing precision while for the electric dipole moment there exists only the upper boundary at a very low limit, $d < 10^{-27}$ e·cm?

c. Determine the orbital momentum and parity quantum numbers if the wave function of a particle has the form ($r^2 = x^2 + y^2 + z^2$)

$$\psi(x, y, z) = (x^2 - y^2)z\, e^{-r^2/a^2}. \qquad (25)$$

d. What is the spin wave function (expressed with respect to the z-axis as quantization axis) for a particle of spin $1/2$ polarized in the direction that is characterized by the polar angle θ and azimuthal angle φ?

e. The scattering cross section for a relativistic electron off the Coulomb potential differs from the Rutherford cross section by the factor

$$1 - (v^2/c^2)\sin^2(\theta/2),$$

where v is the electron velocity and θ scattering angle. For $v \to c$, why does the cross section vanish for backwards scattering, $\theta \to 180°$?