QUANTUM MECHANICS SUBJECT EXAM
Spring 2009

ID NUMBER:______________________________
IMPORTANT: for you own sake, please try not to leave any question blank. If you can't remember a crucial element, write in words what you would do if you had it. Or use another approach, but try to show us what you do know on the topic in question.

1. The DC Stark effect:

   A hydrogen atom is placed in a uniform electric field of strength E_0, whose direction defines the z-axis. Assume that the Stark shift will be large compared to the fine-structure shifts, so that you can perturb the bare hydrogen energy levels, and ignore spin. This problem will focus on the Stark splitting of the $n = 2$ energy level.

   a.) What is the bare energy of the $n = 2$ level?

   b.) Give the degeneracy of the $n = 2$ level, and list the corresponding bare $|n, \ell, m\rangle$ states.

   c.) Recall that the dipole moment operator is $\vec{D} = e\vec{R}$, and write the operator for the interaction energy between the atom and the field.

   d.) In the $n = 2$ subspace, this operator becomes a 4×4 matrix. Determine which matrix elements are non-zero by using $[L_z, Z] = 0$.

   e.) Compute the energy shifts to first-order in E_0. You may leave your answer in terms of a single matrix element, which you do not need to calculate.

2. Two identical fermions in a box:

   Consider two particles (with coordinates x_1 and x_2) in a one-dimensional infinite square-well potential of length L (so that $V(x) = 0$ for $0 < x < L$, and is infinity elsewhere). Consider the two-particle state, $|m, n\rangle$, where one particle is in the m^{th} orbital, and the other is in the n^{th} orbital.

   a.) What is the energy and wavefunction of the state $|m, n\rangle$?

   b.) Assume that the two particles are identical spin-1/2 fermions. If they are in the spin singlet state, what is the ground-state? What is the ground-state energy?

   c.) If they are in the spin triplet state, what is the ground state? What is the ground-state energy?

   e.) For each case, add a particle-particle interaction energy described by the potential $V(x_1, x_2) = g\delta(x_1 - x_2)$. To first-order in g, the energy shift of the ground state, $\Delta E_g = \langle V(X_1 - X_2)\rangle$. Compute the energy shift for both of the above cases. Is the interaction energy larger for the triplet or singlet state?

   Useful formula: $\int_0^L dx\, \sin^2(\frac{m\pi x}{L}) \sin^2(\frac{n\pi x}{L}) = \frac{L}{4}$.

3. The bound state of two nucleons:

The Hamiltonian for the deuteron, a bound state of the proton and neutron, may be written in the form

$$H = \frac{P_p^2}{2M_p} + \frac{P_n^2}{2M_n} + V_1(R) + V_2(R)\vec{S}_p \cdot \vec{S}_n,$$

where $P_{p,n}$ and $M_{p,n}$ are the momentum operators and masses of the proton and neutron, $R = |\vec{R}_p - \vec{R}_n|$ is their relative separation, and $\vec{S}_{p,n}$ are their spin operators. Both are spin-1/2 particles, but they are not identical.

a.) What values are allowed for the total spin angular momentum?

b.) Give 10 distinct quantum numbers that can be assigned to an eigenstate of H? (Note: this total includes s_1 and s_2 even though they are fixed.)

c.) Assuming that the center-of-mass momentum is zero, what one-dimensional wave equation would you have to solve to find the energy eigenvalue associated with one of these eigenstates?

4. Spin 1/2 precession in an oscillating magnetic field:

   The Hamiltonian for an electron (at rest) in a magnetic field is

$$H = -\gamma \vec{B} \cdot \vec{S},$$

   where γ is a constant, $\vec{B}$ is the magnetic field, and $\vec{S}$ is the electron spin operator. Consider the case of an oscillating magnetic field:

$$\vec{B} = B_0 \cos(\omega t) \vec{e}_z,$$

   where B_0 and ω are constants, and $\vec{e}_z$ is the unit vector along the z-axis.

   a.) Write down this Hamiltonian as a 2×2 matrix in the basis of S_z eigenstates.

   b.) Write down the time-dependent Schrödinger equation for this Hamiltonian.

   c.) Find the general solution of the Schrödinger equation for an arbitrary initial state of the form $|\psi(0)\rangle = a|\uparrow_z\rangle + b|\downarrow_z\rangle$.

   d.) If the particle is in the spin-up state with respect to the x-axis, $|\uparrow_x\rangle$, at $t = 0$, what is the probability to find the electron in state $|\uparrow_x\rangle$ at some later time $t > 0$.

   e.) Assuming that the measurement was performed, and result $|\uparrow_x\rangle$ was obtained, what was the state of the system immediately after the measurement?

5. General scattering theory:

   a.) How is the differential cross-section $\frac{d\sigma(k)}{d\Omega}$ related to the scattering amplitude $f(\theta, \phi|k)$?

   b.) How is the total cross-section, $\sigma_{tot}(k)$ related to the differential cross-section?

   Partial- wave scattering:

   Consider a soft-sphere scatterer, defined by

$$V(r) = \begin{cases} U_0 & r < r_0 \\ 0 & r > r_0 \end{cases}$$

   c.) For the case $k < \sqrt{2MU_0}/\hbar$ (i.e. the particle tunnels into the sphere), compute the s-wave scattering phase shift. Check carefully that you satisfy the correct boundary condition at $r = 0$.

   d.) Verify that you get the result you expect in the limit $U_0 \to \infty$.

6. A system is described by the Hamiltonian

$$H = -\frac{\hbar^2}{4m\lambda^2}\left(AA + A^\dagger A^\dagger - AA^\dagger - A^\dagger A\right),$$

where A and $A^\dagger$ are the harmonic oscillator lowering and raising operators, which satisfy

$$[A, A^\dagger] = 1.$$

a.) Derive the Heisenberg equation of motion for A_H.

b.) Based on your answer to a.), what is the equation of motion for $A_H^\dagger$?

c.) From these two equations, compute $\frac{d}{dt}X_H$, where $X_H = \frac{\lambda}{\sqrt{2}}\left(A_H + A_H^\dagger\right)$; then re-express your answer in terms of $P_H = -i\frac{\hbar}{\sqrt{2}\lambda}\left(A_H - A_H^\dagger\right)$.

d.) Using your answers to parts (a), (b) compute $\frac{d}{dt}P_H$. Find the solution to this equation, then use the solution for $P_H(t)$ to find the solution to your equation for $X_H(t)$.

(Hint: your answers should give expressions for $X_H(t)$ and $P_H(t)$ in terms of their initial values.)