

**PHY831 - Subject Exam Dec. 14th 2011, 10am - 1pm**

Answer all questions (total points available is 100). Time for exam - 3 hours

Name:

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \left(\frac{\pi}{a}\right)^{1/2} e^{\frac{b^2}{4a}}; \quad \int_0^{\infty} dx x^n e^{-ax^2} = \frac{1}{2a^{(n+1)/2}} \Gamma\left(\frac{n+1}{2}\right) \quad (1)$$

$$\int \frac{dx}{(1+x^2)^{1/2}} = \text{Sinh}^{-1}(x); \quad \int_0^{\infty} \frac{x^{s-1} dx}{e^x - 1} = \Gamma(s)\zeta(s), \quad (2)$$

where  $\Gamma(s) = (s-1)!$  for  $s$  a positive integer, and  $\zeta(2) = \frac{\pi^2}{6}$ ,  $\zeta(3) = 1.202\dots$ ,  $\zeta(4) = \pi^4/90$ .

$$\left(\frac{\partial x}{\partial y}\right)_z = 1/\left(\frac{\partial y}{\partial x}\right)_z; \quad \left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial w}\right)_z \left(\frac{\partial w}{\partial y}\right)_z; \quad (3)$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1; \quad \left(\frac{\partial A}{\partial x}\right)_z = \left(\frac{\partial A}{\partial x}\right)_y + \left(\frac{\partial A}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z; \quad (4)$$

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_{V,N} = T \left(\frac{\partial S}{\partial T}\right)_{V,N}; \quad C_P = \left(\frac{\partial H}{\partial T}\right)_{P,N} = T \left(\frac{\partial S}{\partial T}\right)_{P,N}, \quad (5)$$

$$\kappa_S = -\frac{1}{V} \left(\frac{\partial V}{\partial P}\right)_{S,N}; \quad \kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial P}\right)_{T,N}; \quad \alpha_P = \frac{1}{V} \left(\frac{\partial V}{\partial T}\right)_{P,N} \quad (6)$$

$$\frac{U}{L^d} = \frac{c_d}{h^d} \int_0^{\infty} dp p^{d-1} (cp)^s \frac{ze^{-\beta cp^s}}{1 - ze^{-\beta cp^s}} = \frac{d k_B T}{s \lambda^d} g_{1+d/s}(z); \quad P = \frac{s U}{d L^d} \quad (7)$$

$$I(r, \nu \rightarrow \infty) = \int_0^{\infty} dx x^{r-1} \frac{ze^{-x}}{1+ze^{-x}} = \frac{1}{r} \int_0^{\infty} dx x^r \frac{e^{-x-\nu}}{(e^{x-\nu}+1)^2} = \frac{1}{r} [\nu^r + \frac{\pi^2}{6} r(r-1)\nu^{r-2} + \dots] \quad (8)$$

$$H = -\frac{1}{2} \sum_{ij} J_{ij} S_i S_j - \sum_i h_i S_i \rightarrow (\text{for spin } \pm 1) Z_{MF} = 2^N e^{-\frac{1}{2}\beta \sum_{ij} J_{ij} m_i m_j} \prod_{i=1}^N (\text{Cosh}(\beta \sum_j J_{ij} m_j + \beta h_i)). \quad (9)$$

$$Z_{vdw} = \frac{q^N}{N! \lambda^{3N}}; \quad \text{where } q = (V - Nb) e^{aN/(V k_B T)}; \quad \frac{Pv}{k_B T} = \sum_{l=1}^{\infty} a_l(T) \left(\frac{\lambda^3}{v}\right)^{l-1} \quad (\text{virial expansion}) \quad (10)$$

$$g_{GL} = \int dV \left[ \frac{1}{2m} |(-i\hbar \nabla - qA)\psi(\vec{r})|^2 + a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2} |\psi(\vec{r})|^4 + \frac{B^2}{2\mu_0} + \frac{\mu_0 H^2}{2} - B \cdot H \right]. \quad (11)$$

$$\frac{\delta g}{\delta \psi^*} = a(T)\psi + b(T)|\psi|^2\psi + \frac{1}{2m} (-i\hbar \nabla - qA)^2 \psi = 0; \quad (12)$$

$$\frac{\delta g}{\delta A} = 0 \rightarrow j_s = \frac{-iq\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{q^2}{m} A |\psi|^2 = \frac{q}{m} |\psi|^2 (\hbar \nabla S - qA) = q |\psi|^2 v_s \quad (13)$$

$$H_{MF} - \mu N = \sum_{\vec{k}\sigma} (\epsilon_{\vec{k}\sigma} - \mu) a_{\vec{k}\sigma}^\dagger a_{\vec{k}\sigma} - \sum_{\vec{k}} (\Delta_{\vec{k}} a_{\vec{k}\uparrow}^\dagger a_{-\vec{k},\downarrow}^\dagger + \Delta_{\vec{k}}^* a_{-\vec{k},\downarrow} a_{\vec{k}\uparrow} - b_{\vec{k}}^* \Delta_{\vec{k}}); \quad (14)$$

$$H_{MF} - \mu N = \sum_{\vec{k}} (\epsilon_{\vec{k}} - \mu - E_{\vec{k}} + \Delta_{\vec{k}} b_{\vec{k}}^*) + \sum_{\vec{k}} E_{\vec{k}} (\gamma_{\vec{k}\uparrow}^\dagger \gamma_{\vec{k}\uparrow} + \gamma_{\vec{k}\downarrow}^\dagger \gamma_{\vec{k}\downarrow}), \quad (15)$$

$$\Delta_{\vec{k}} = - \sum_{\vec{l}} V_{\vec{k}\vec{l}} b_{\vec{l}}; \quad b_{\vec{k}} = \langle a_{-\vec{k},\downarrow} a_{\vec{k}\uparrow} \rangle = \frac{\Delta_{\vec{k}}}{2E_{\vec{k}}} (1 - 2f(E_{\vec{k}})); \quad E_{\vec{k}} = ((\epsilon_{\vec{k}} - \mu)^2 + |\Delta_{\vec{k}}|^2)^{1/2} \quad (16)$$

**Problem 1. (16 points)**

(i) (10 points) Show that the fluctuations of the energy  $E$  in the canonical ensemble are given by,

$$\delta E^2 = \langle E^2 \rangle - \langle E \rangle^2 = k_B T^2 C_V. \quad (17)$$

where  $\langle O \rangle$  indicates the ensemble average of the operator  $O$ ,  $C_V$  is the heat capacity at constant volume,  $k_B$  is Boltzmann's constant and  $T$  is temperature.

(ii) (6 points) Using this result explain why  $\delta E / \langle E \rangle$  is proportional  $1/N^{1/2}$ , where  $N$  is the number of particles or spins in the system. What is the significance of this result for the connections between statistical physics and thermodynamics?

**Problem 2. (16 points)**

An ideal monatomic Bose gas of particles of mass  $m$  has dispersion relation  $\epsilon_p = cp^s$ , where  $c$  is a constant (e.g. for a non-relativistic gas  $c = \hbar^2/2m$ ,  $s = 2$ ), so that its thermodynamics in  $d$  dimensions is described by,

$$\frac{U}{L^d} = \frac{d k_B T}{s \lambda^d} g_{1+d/s}(z); \quad \frac{N}{L^d} = \frac{1}{\lambda^d} g_{d/s}(z) + \frac{1}{L^d} \frac{z}{1-z}; \quad P = \frac{s U}{d L^d}; \quad g_y(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^y} \quad (18)$$

where  $N$  is the number of particles and  $L^d$  is the size of the hypercubic container in  $d$  dimensions. The fugacity  $z = e^{\beta\mu}$ , where  $\mu$  is the chemical potential.

(i) (5 points) Find the critical dimension  $d_c$  as a function of  $s$  above which there is a Bose condensation phase transition at finite temperature.

(ii) (6 points) For cases where there is a finite temperature Bose condensation transition, find an expression for the order parameter (the Bose condensate fraction  $N_0/N$ ) as a function of temperature, where  $N_0$  is the number of particles in the ground state.

(iii) (5 points) Find the critical exponent describing the scaling behavior of the order parameter near the critical point. Does the order parameter critical exponent depend on the spatial dimension? Is this consistent with the universality hypothesis?

**Problem 3. (16 points)**

(i) (5 points) On a  $P-v$  diagram, where  $v = V/N$ , plot the isotherms of the van der Waals equation. Also sketch the spinodal lines which are defined by  $\partial P / \partial v = 0$ . Here  $P$  is pressure,  $V$  is volume and  $N$  is the number of particles in the system. Sketch the boundaries of the co-existence region and state the Maxwell condition used to construct them.

(ii) (8 points) Using the fact that the liquid gas transition is in the same universality class as the Ising model, find the upper and low critical dimensions for the liquid-gas phase transition.

(iii) (3 points) What is the order parameter for the liquid-gas phase transition - give an expression for the expected scaling behavior of the order parameter near the critical point of the liquid-gas system. What is the mean field value of the order parameter exponent?

**Problem 4. (16 points)**

Consider  $N$  electrons moving in a two dimensional plane of area  $A = L^2$ . A magnetic field,  $B$ , is applied along the normal to the plane. The Bohr magneton is defined as  $\mu_B = e\hbar/(2m_e)$ . The magnetic moment of the electron is, to a good approximation, equal to  $\mu_B$ .

(i) (3 points) Find the lowest magnetic field  $B_1$  that is strong enough so that all of the electrons can be accommodated in the lowest Landau level (Hint: the degeneracy of a spinless Landau level is  $g = BAe/h$ , where  $h$  is Planck's constant).

(ii) (3 points) Find the field  $B_2$  above which all of the electrons are in the up spin sub-level of the first Landau level.

(iii) (10 points) Taking into account both the paramagnetic and diamagnetic contributions, find an expression for the magnetization of the free electron gas at temperature  $T = 0K$  as a function of the magnetic field for  $B > B_1$ . Sketch the behavior of the magnetization as a function of the field.

**Problem 5. (18 points)**

(i) (8 points) (a) Sketch the field-temperature phase diagram for a type II superconductor indicating the Meissner, mixed and normal phases; (b) Sketch the behavior of the superconducting gap,  $\Delta$ , as a function of temperature, at zero applied magnetic field, for an s-wave BCS superconductor. (c) Write down a mathematical expression for the scaling behavior of the gap near the critical temperature,  $T_c$ ; (d) For an s-wave BCS superconductor, sketch the density of states of the quasiparticle excitations from the BCS ground state as a function of energy, near the Fermi energy,  $\epsilon_F$ .

(ii) (4 points) Within London theory, which is valid in the extreme type II limit, the Helmholtz free energy per unit length of an isolated flux quantum in a type II superconductor is approximately,

$$f_1 \approx \frac{\phi_0^2}{4\pi\mu_0\lambda^2} \text{Ln}\left(\frac{\lambda}{\xi}\right) \quad (19)$$

Using this result find an expression for the lower critical field  $H_{c1}$  of a type II superconductor. Here  $\phi_0$  is the flux quantum,  $\lambda$  is the penetration depth,  $\xi$  is the coherence length,  $\mu_0$  is the permeability.

(iii) (6 points) By linearizing the G-L equation, find an expression for the upper critical field of the superconductor,  $H_{c2}$ , in terms of the parameters in the G-L theory. To write the result in its usual form you can use  $\xi^2 = \hbar^2/(2m|a|)$ ,  $\phi_0 = h/q = h/2e$ . (hint: the eigenvalues of a Cooper pair in a magnetic field are,  $\epsilon_{n,k_z} = (n + 1/2)\hbar\omega_c + \hbar^2 k_z^2/2m$ )

**Problem 6. (18 points)**

A spin half Ising model with three spin interactions on a triangular lattice has Hamiltonian,

$$H = - \sum_{ijk \in \Delta} JS_i S_j S_k \quad (20)$$

where the sum is over all nearest neighbor triangles on an infinite triangular lattice, and the interaction is ferromagnetic  $J > 0$ .

(i) (6 points) Using a leading order expansion in the fluctuations (i.e. write  $S_i = m_i + (S_i - m_i)$  and expand to leading order in the fluctuations), find the mean field Hamiltonian for this problem (Here  $m_i = \langle S_i \rangle$  is the magnetization at site  $i$ ).

(ii) (6 points) Using the mean field Hamiltonian and assuming a homogeneous state where  $m_i = m$ , find an expression for the mean field Helmholtz free energy, and the mean field equation for this problem.

(iii) (6 points) Taking  $J = 1$ , sketch the behavior of the solutions to the mean field equation as a function of temperature. Does the non-trivial solution move continuously toward the  $m = 0$  solution as the temperature increases? Is the behavior of the order parameter at the critical temperature discontinuous or continuous? Do you expect the correlation length to diverge at the critical point in this problem?