

Classical Mechanics Graduate Exam
August 29, 2006

NOTE: Do NOT write your name on any paper you hand in. Only write your selected number for subsequent identification, so that the problems may be graded anonymously. Each problem will be handed in separately, so your number and the problem number should be given on the first page of your work for each problem.

Show all work. Justify your answers.

1. (10 points) In Galileo's famous experiment, two balls of differing radii were dropped from the tower of Pisa. Suppose the balls were iron spheres of density $\rho_{\text{Fe}} = 7500 \text{ kg/m}^3$ and radii of 1 cm and 5 cm. In addition to the gravitational force, the balls experience a frictional force $F_{\text{fric}} = C(rv)^2$, where v is the speed of the ball, r is its radius and $C = 1 \text{ kg/m}^3$. Assume the balls were dropped simultaneously from a height of 15 m.

- (a) (2 pt) Calculate the terminal velocities, v_t , of the two balls in m/s.
- (b) (4 pt) Calculate the velocity of a ball of radius r as a function of time including the frictional force. Note:

$$\int dx \frac{1}{x^2 - 1} = -\tanh^{-1}(x).$$

- (c) (2 pt) Find an approximate expression for the time required for a ball of radius r to reach a speed of $0.99v_t$.
- (d) (2 pt) Which ball reaches the ground first? State your reason.

2. (10 points) The function $y(x)$ which minimizes a variational integral of the form

$$I = \int_{x_0}^{x_1} dx F(y(x), y'(x))$$

where the prime denotes a derivative with respect to x , satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0.$$

(a) (3 pt) Using the equations of motion, show that the function H defined by

$$H = y' \frac{\partial F}{\partial y'} - F$$

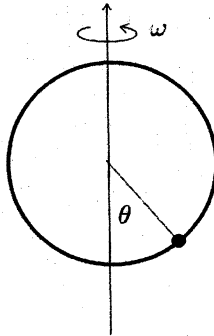
satisfies

$$\frac{dH}{dx} = 0 \quad \text{or} \quad H = \text{constant}.$$

(b) (7 pt) Using H , find the function $y(x)$ which connects the points $(1, -1)$ and $(1, 1)$ and minimizes the integral

$$I = \int_{-1}^1 dx \sqrt{y} \sqrt{1 + y'^2}.$$

3. (10 points) A bead of mass m is constrained to move on a circular hoop of radius R . The hoop rotates with constant angular velocity ω around a fixed vertical diameter of the hoop, parallel to the uniform gravitational field.
- (a) (2 pt) Calculate the Lagrangian for this system in appropriate variables.
 - (b) (1 pt) Derive Lagrange's equations of motion.
 - (c) (2 pt) Derive the Hamiltonian function of canonical variables from its definition in terms of the Lagrangian (explain the steps).
 - (d) (1 pt) Derive Hamilton's equation of motion for the canonical variables.
 - (e) (2 pt) Find the critical angular velocity Ω below which the bottom of the hoop is a stable equilibrium position for the bead.
 - (f) (2 pt) Find the stable equilibrium position for $\omega > \Omega$.



4. (10 points) Consider the Lagrangian

$$L = (m/2)([dx/dt]^2 - \omega^2 x^2)e^{\gamma t}$$

for motion in the x direction of a particle of mass m .

- (a) (2 pt) Derive Lagrange's equation of motion.
- (b) (1 pt) Interpret the equation of motion by stating what kinds of force are acting on the particle.
- (c) (2 pt) Calculate the canonical momentum, and from this construct the Hamiltonian function defined in terms of the Lagrangian.
- (d) (2 pt)
 - i. Is the Hamiltonian a constant of the motion?
 - ii. Is the energy conserved?
 - iii. Is the frictional force doing positive or negative work on the particle?
 - iv. Explain your answers to i), ii) and iii) in words.
- (e) (3 pt) For the initial conditions $x(0) = 0$ and $(dx/dt)|_{t=0} = v_0$, what is $x(t)$ asymptotically as $t \rightarrow \infty$. Find all possible solutions, depending on how the relative magnitudes of γ and ω affect the motion.

5. (10 points) Discuss the 2-dimensional motion of a particle moving in an attractive central-force described by the force law $f(r) = -k/r^\alpha$, where k is positive and $3 > \alpha > 2$.
- (a) (4 pt) Write down the equations of motion in polar coordinates.
 - (b) (4 pt) Show how conservation laws can be used to derive the formal equation for the orbit of motion.
 - (c) (2 pt) Describe the nature of the orbits for various possible initial energies and angular momenta using a graph of energy vs r .