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MICHIGAN STATE UNIVERSITY

PHY422/820: Classical Mechanics

Subject Exam

December 12, 2021

Student Number: STUDNUMBER

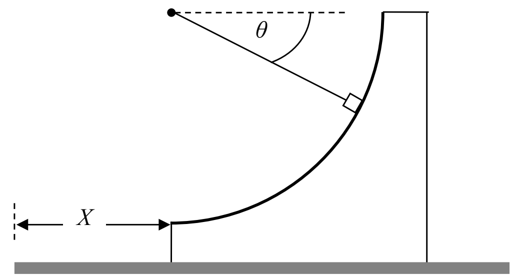
Points					
F1	F2	F3	F4	F5	total
10	10	10	10	10	50

- **This is a closed-notes exam: You are not allowed to use the text books, lecture materials or external resources. You are not allowed to discuss this exam or questions related to the exam with your fellow students or with third parties during the exam time window.**
- **Take note of the included formula sheet.**
- **Read through the whole exam before starting to work.**
- **Not all questions are equally difficult, and you may wish to start with problems that play to your strengths.**
- **In some problems, intermediate results are provided as a check and a means to continue working on later parts if you are stuck.**
- **Document all your work (including scratch paper!) so that you can receive partial credit. Justify all your answers!**
- **Do not hesitate to ask if anything is unclear!**

Good Luck!

Problem F1 – Moving Circular Wedge

[10 Points] A block of mass m is sliding down a smooth circular wedge of mass M, which can itself move freely on a smooth horizontal plane.



1. Construct the Lagrangian of the system, using the general coordinates X and θ as indicated in the figure.

2. Determine the Lagrange equations for X and θ.

3. Identify at least two conserved quantities and (briefly) state the physical reason for the conservation law.

4. Use the conservation law(s) to eliminate X and derive a single equation of motion in the coordinate θ. (You do not have to solve this equation.)

Problem F2 – A Central Force

[10 Points] An object of mass m moves under the influence of the potential

$$V(r) = V_0 \ln\left(\frac{r}{r_0}\right) , \tag{1}$$

where V_0 and $r_0 > 0$ are constants.

1. What is the central force $F(r)$ acting on the object?

2. Are bounded orbits possible? How does the answer depend on the energy E of the orbit and on the sign of V_0? Sketch $V(r)$ and $V_{\text{eff}}(r)$ for both possible signs of V_0. Indicate the different kinds of *allowed* trajectories (circular, bound orbit, scattering, ...) by appropriate lines with fixed E in your figures.

   HINT: Make sure to carefully consider the behavior of $V(r)$ at large distances.

3. Consider a situation where circular orbits are possible. How is the radius R of a circular orbit related to its angular momentum l?

4. Examine the stability of the circular orbits you found. Find the frequency of oscillations for near-circular orbits.

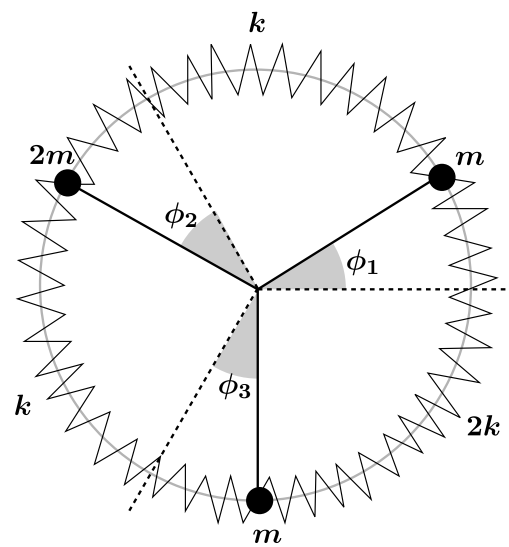
Problem F3 – Normal Modes

[10 Points] Three masses that are connected by springs are moving without friction on a circular horizontal track of radius R. The sizes of the masses and strengths of the spring constants are indicated in the figure.

1. Construct the Lagrangian for the system in terms of the displacements from equilibrium ϕ_1, ϕ_2, ϕ_3.

   Hint: Distances on the circular track can be expressed in terms of arc lengths if necessary.

2. Determine the normal modes, i.e., characteristic frequencies and vectors (vectors do not need to be normalized). Sketch and interpret your solutions.



Problem F4 – Rotating Cube

[10 Points] The density distribution of a solid homegenous cube of mass M with side length a can be expressed with the help of Heaviside step functions Θ (see formula sheet) as

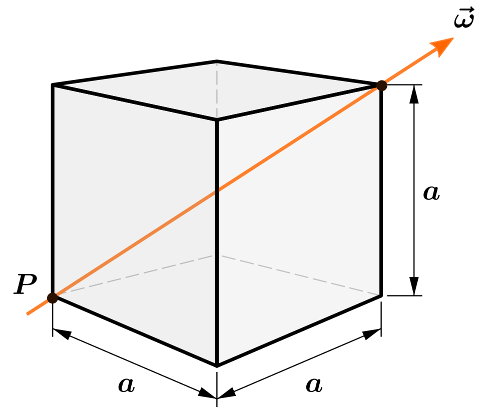
$$\rho(x,y,z) = \frac{M}{a^3}\Theta\left(\frac{a}{2}-|x|\right)\Theta\left(\frac{a}{2}-|y|\right)\Theta\left(\frac{a}{2}-|z|\right) \tag{2}$$

where we have chosen the cube's **center of mass as the origin** of the coordinate system.

1. Compute the volume integral of the density $\rho(x,y,z)$ to validate that it yields the total mass of the cube.

2. Construct the moment of inertia tensor with respect to the center of mass and determine the principal axes. **Note: Prior knowledge can only be used to validate entries of *I*.**

3. In the CoM system, the cube's front lower left corner (indicated by the point P in the figure) has the coordinates $\vec{R} = \left(-\frac{a}{2}, -\frac{a}{2}, -\frac{a}{2}\right)^T$. Compute the moment of inertia tensor with respect to this point.



4. The cube is rotating around the diagonal through the corner P and its opposite with an angular velocity vector $\vec{\omega} = \omega_0\,(1,1,1)^T$ (see figure). Compute the angular momentum and the rotational energy.

Problem F5 – Hamiltonian Treatment of the Kepler Problem

[10 Points] Consider a mass m that is moving in the Kepler potential

$$V(r) = -\frac{\kappa}{r}\,, \quad \kappa > 0\,. \tag{3}$$

In the following, we will work in **Cartesian coordinates** (x, y), so that $r = \sqrt{x^2 + y^2}$.

1. Determine the canonical momenta (p_x, p_y) and construct the Hamiltonian in the canonical variables.

2. Use the properties of the Hamiltonian and Poisson brackets to show that the total energy and the angular momentum $l = |\vec{l}| = xp_y - yp_x$ are conserved.
   HINT: It is helpful to evaluate $\frac{\partial}{\partial x} f(r)$ and $\{p_x, f(r)\}$ first.

3. Consider
   $$A = p_y - \beta \frac{x}{r} \tag{4}$$
   and compute its Poisson bracket with H. Determine the values β for which A is conserved, and relate these values to the constants of the problem.
   HINT: Remember the conservation laws and $r^2 = x^2 + y^2$.