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Electricity & Magnetism **Subject Exam** **January 14, 2015**

Please read all of the following before starting the exam:

- Before starting the exam, write your student number on each page of the exam. If you require extra paper, write your student number and the relevant problem number on the extra page(s).
- All problems are assumed to be in Gaussian units. If you choose to convert to SI units, please state so. You will be responsible for the correct conversion factors.
- You may use a simple calculator, but no external notes, books, etc.
- Show all work as neatly and logically as possible to maximize your credit. Circle or otherwise indicate your final answers.
- This test has 5 problems for a total of 100 points. Please make sure that you have all of the pages.
- Good luck!

VECTOR CALCULUS

Gradient vector $= \vec{\nabla} = (\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$
div $\mathbf{v} \equiv \vec{\nabla} \cdot \mathbf{v}$
curl $\mathbf{v} \equiv \vec{\nabla} \times \mathbf{v}$

Cylindrical coordinates (ρ, ϕ, z):

$$\nabla S = \frac{\partial S}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial S}{\partial \phi}\mathbf{e}_\phi + \frac{\partial S}{\partial z}\mathbf{e}_z$$

$$\vec{\nabla} \cdot \mathbf{v} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho v_\rho) + \frac{1}{\rho}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\begin{aligned}\vec{\nabla} \times \mathbf{v} &= \left[\frac{1}{\rho}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\mathbf{e}_\rho \\ &\quad + \left[\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho}\right]\mathbf{e}_\phi \\ &\quad + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho v_\phi) - \frac{\partial v_\rho}{\partial \phi}\right]\mathbf{e}_z\end{aligned}$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical polar coordinates (r, θ, ϕ):

$$\nabla S = \frac{\partial S}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial S}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}\mathbf{e}_\phi$$

$$\vec{\nabla} \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned}\vec{\nabla} \times \mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\mathbf{e}_r \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\mathbf{e}_\theta \\ &\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\mathbf{e}_\phi\end{aligned}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Spherical Harmonics:

$$l = 0: \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l = 1: \qquad Y_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$l = 2: \qquad Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}$$

$$Y_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

Legendre Polynomials:

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l,0}(\theta,\phi)\ .$$

SELECTED NUMERICAL DATA

Speed of light $c = 3 \times 10^{10}\,\mathrm{cm/s}$,
Elementary charge $e = 4.8 \times 10^{-10}$ statC,
Planck constant $\hbar = h/2\pi = 1.055 \times 10^{-27}\,\mathrm{erg \cdot s} = 6.582 \times 10^{-22}\,\mathrm{MeV \cdot s}$.
Do not use these numbers directly! Instead combine your expressions in the standard combinations:
Fine structure constant (dimensionless) $\alpha = e^2/\hbar c$, $1/\alpha = 137.036$;
$\hbar c = 197.3\,\mathrm{MeV \cdot fm} \approx 2 \times 10^{-5}\,\mathrm{eV \cdot cm}$ $(1\,\mathrm{fm} = 10^{-13}\,\mathrm{cm})$.

Electron mass $m = 0.911 \times 10^{-27}\,\mathrm{g} = 0.511\,\mathrm{MeV}/c^2$,
Proton mass $m_p = 1.673 \times 10^{-24}\,\mathrm{g} = 938.3\,\mathrm{MeV}/c^2 = 1836.2\,m$,

Compton wave length of the electron $\lambda_e = \hbar/mc = 3.862 \times 10^{-11}\,\mathrm{cm}$,
Classical electron radius $r_e = e^2/mc^2 = 2.818 \times 10^{-13}\,\mathrm{cm}$.

Bohr magneton $\mu_B = e\hbar/2mc = 9.274 \times 10^{-21}\,\mathrm{erg/Gs}$,
Nuclear magneton (n.m.) $\mu_N = e\hbar/2m_pc = \mu_B(m/m_p) = 5.051 \times 10^{-24}\,\mathrm{erg/Gs}$,
Proton magnetic moment $\mu_p = 2.793$ n.m.,
Neutron magnetic moment $\mu_n = -1.913$ n.m.

Gravitational constant $G = 6.67 \times 10^{-8}\,\mathrm{cm^3 g^{-1} s^{-2}}$.

Some conversions between SI and Gaussian units:

$$1\,\mathrm{J} = 10^7\,\mathrm{erg}$$

$$1\,\mathrm{Watt} = 10^7\,\mathrm{erg/s}$$

$$1\,\mathrm{Coulomb} = 3 \times 10^9\,\mathrm{statC}$$

$$1\,\mathrm{Ampere} = 3 \times 10^9\,\mathrm{statA}$$

$$1\,\mathrm{Volt} = \frac{1}{300}\,\mathrm{statV}$$

$$1\,\mathrm{Tesla} = 10^4\,\mathrm{Gs}$$

$$1\,\mathrm{eV}/c^2 = 1.783 \times 10^{-36}\,\mathrm{kg} = 1.783 \times 10^{-33}\,\mathrm{g}$$

Conversion of Maxwell Equations from Gaussian to SI units:

$$(\rho, \mathbf{j}, q) \Rightarrow \frac{(\rho, \mathbf{j}, q)}{\sqrt{4\pi\epsilon_0}},$$

$$(\phi, \mathbf{E}) \Rightarrow \sqrt{4\pi\epsilon_0}\,(\phi, \mathbf{E}),$$

$$(\mathbf{A}, \mathbf{B}) \Rightarrow \sqrt{\frac{4\pi}{\mu_0}}\,(\mathbf{A}, \mathbf{B}),$$

$$c = \sqrt{\frac{1}{\epsilon_0\mu_0}}.$$

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1. A spherical shell of radius R has a surface charge density of $\sigma(\theta, \phi) = \sigma_0 \sin^2\theta$, where $\{r, \theta, \phi\}$ are spherical coordinates.

   (a) [**16 pts**] Find the potential $\Phi(r, \theta, \phi)$ for both $r < R$ and $r > R$.

   (b) [**4 pts**] Identify the terms in the solution for $r > R$ in terms of multipole moments.

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2. A cylindrical conductor of radius a has a cylindrical hole of radius b bored parallel to and centered a distance d from the cylinder axis ($d + b < a$). The current density $\mathbf{j}$ is uniform throughout the remaining metal of the cylinder and is parallel to the axis of the cylinder. The net electric charge on the cylinder is zero.

   (a) [**15 pts**] Find the magnitude and direction of the magnetic field at an arbitrary point inside the hole. To be specific, let the current run in the positive z direction ($\mathbf{j} = j\hat{\mathbf{z}}$), let the center of the conductor pass through the origin, and let the center of the cylindrical hole pass through the point $\mathbf{d} = d\hat{\mathbf{x}}$.

   (b) [**5 pts**] If this system is observed in a reference frame moving at a constant velocity $\boldsymbol{\beta} = \beta\hat{\mathbf{z}}$ what will be the electric and magnetic fields in the hole?

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3. A small static magnetic moment $\boldsymbol{\mu}$ and a static electric charge e are placed at the origin.

   (a) **[10 pts]** Calculate the magnitude and the direction of the Poynting vector of the electromagnetic field created by this system.

   (b) **[10 pts]** Calculate the energy flux through a spherical surface enclosing the origin at its center.

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4. **[20 pts]** A photon is scattered off of a proton, that is initially at rest, and produces an outgoing proton and an outgoing neutral pion ($\gamma + p \rightarrow p + \pi^0$). [Data: $m_p = 938.3$ MeV/c^2, $m_{\pi^0} = 135.0$ MeV/c^2.]

   (a) **[15 pts]** What is the minimum initial energy that the photon must have in order for this process to occur?

   (b) **[5 pts]** If the photon has this minimum initial energy, what will be the velocity of the neutral pion in the lab frame? [HINT: In order to reduce tedious algebra, consider what is special about the velocities of the outgoing particles when they are produced at threshold.]

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5. An antenna is in the shape of a circular loop of radius a. The loop carries a current given by $I = I_0 \cos \omega t$.

   (a) [**10 pts**] Obtain the angular distribution of the power, $dP/d\Omega$, averaged over one period, assuming that the size of the antenna is much smaller than the wavelength of the radiation.

   (b) [**10 pts**] Obtain the total power, P, emitted by the antenna, averaged over one period.